

AN INTEGRAL IDENTITY WITH APPLICATIONS IN ORTHOGONAL POLYNOMIALS

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ABSTRACT. For $\lambda = (\lambda_1, \dots, \lambda_d)$ with $\lambda_i > 0$, it is proved that

$$\prod_{i=1}^d \frac{1}{(1 - rx_i)^{\lambda_i}} = \frac{\Gamma(|\lambda|)}{\prod_{i=1}^d \Gamma(\lambda_i)} \int_{\mathcal{T}^d} \frac{1}{(1 - r\langle x, u \rangle)^{|\lambda|}} \prod_{i=1}^d u_i^{\lambda_i - 1} du,$$

where \mathcal{T}^d is the simplex in homogeneous coordinates of \mathbb{R}^d , from which a new integral relation for Gegenbauer polynomials of different indexes is deduced. The latter result is used to derive closed formulas for reproducing kernels of orthogonal polynomials on the unit cube and on the unit ball.

1. INTRODUCTION

Let $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x_1 \geq 0, \dots, x_d \geq 0\}$ be the positive quadrant of \mathbb{R}^d . Let T^{d-1} be the simplex in \mathbb{R}^{d-1} defined by $T^{d-1} := \{y \in \mathbb{R}_+^{d-1} : |y| \leq 1\}$, where $|y| := y_1 + \dots + y_{d-1}$. Written in homogeneous coordinates, this simplex is equivalent to

$$\mathcal{T}^d := \{y \in \mathbb{R}_+^d : |y| = 1\}.$$

Observe that \mathcal{T}^2 reduces to the interval $[0, 1]$. The main result in this paper is the following integral identity and its applications.

Theorem 1.1. *Let $d = 2, 3, \dots$ and $\lambda = (\lambda_1, \dots, \lambda_d)$ with $\lambda_i > 0$, $1 \leq i \leq d$. For $x \in \mathbb{R}^d$ and $r \geq 0$ such that $r|x_i| \leq 1$, $1 \leq i \leq d$,*

$$(1.1) \quad \prod_{i=1}^d \frac{1}{(1 - rx_i)^{\lambda_i}} = \frac{\Gamma(|\lambda|)}{\prod_{i=1}^d \Gamma(\lambda_i)} \int_{\mathcal{T}^d} \frac{1}{(1 - r\langle x, u \rangle)^{|\lambda|}} \prod_{i=1}^d u_i^{\lambda_i - 1} du.$$

Although the identity (1.1) is elementary, it leads to new identities on the Gegenbauer polynomials that have interesting applications to orthogonal polynomials in several variables. For $\lambda > -1/2$, define

$$w_\lambda(t) := (1 - t^2)^{\lambda - 1/2}, \quad -1 < t < 1.$$

The Gegenbauer polynomial C_n^λ is defined as the orthogonal polynomial of degree n with respect to w_λ , normalized by $C_n^\lambda(1) = \binom{n+2\lambda-1}{n}$. It satisfies the relation

$$c_\lambda \int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t) w_\lambda(t) dt = h_n^\lambda \delta_{n,m}, \quad h_n^\lambda := \frac{\lambda}{n + \lambda} C_n^\lambda(1),$$

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where c_λ is the normalization constant of w_λ . Let $\tilde{C}_n^\lambda(t) := C_n^\lambda(t)/\sqrt{h_n^\lambda}$. Then \tilde{C}_n^λ is the n -th orthonormal polynomial with respect to w_λ . It follows readily that

$$Z_n^\lambda(t) := \tilde{C}_n^\lambda(1)\tilde{C}_n^\lambda(t) = \frac{n+\lambda}{\lambda}C_n^\lambda(t).$$

The Gegenbauer polynomials satisfy the following generating relations: for $0 \leq r < 1$,

$$(1.2) \quad \frac{1}{(1-2rt+r^2)^\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t)r^n \quad \text{and} \quad \frac{1-r^2}{(1-2rt+r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} Z_n^\lambda(t)r^n.$$

One application of Theorem 1.1 is a closed formula for the reproducing kernels of product Gegenbauer polynomials on the cube $[-1, 1]^d$, which will be discussed in Subsection 3.2. Another application of Theorem 1.1, more directly, gives two new identities for the Gegenbauer polynomials.

Theorem 1.2. *For $\lambda > -1/2$ and $\mu > 0$,*

$$(1.3) \quad C_n^\lambda(x) = c_\mu \sigma_{\lambda,\mu} \int_{-1}^1 \int_0^1 C_n^{\lambda+\mu}(sx + (1-s)y) s^{\lambda-1} (1-s)^{\mu-1} ds w_\mu(y) dy,$$

and, furthermore,

$$(1.4) \quad Z_n^\lambda(x) = c_\mu \sigma_{\lambda+1,\mu} \int_{-1}^1 \int_0^1 Z_n^{\lambda+\mu}(sx + (1-s)y) s^\lambda (1-s)^{\mu-1} ds w_\mu(y) dy,$$

where

$$\sigma_{\lambda,\mu} := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)\Gamma(\mu)} \quad \text{and} \quad c_\mu := \frac{\Gamma(\mu+1)}{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})}.$$

These identities are new. It is known in the literature (see, for example, [1, p. 25]) that if $\mu > 0$ and $\lambda > -1/2$, then

$$\frac{C_n^{\lambda+\mu}(x)}{C_n^{\lambda+\mu}(1)} = \int_{-1}^1 \frac{C_n^\lambda(y)}{C_n^\lambda(1)} d\mu_x(y), \quad -1 \leq x \leq 1,$$

where $d\mu_x(y)$ is strictly positive and absolutely continuous when $-1 < x < 1$, and it is a unit mass at $y = x$ when $x^2 = 1$. In comparison, the index of the Gegenbauer polynomial in the left hand side of the identity (1.3) is smaller than the index appeared in the right hand side.

The new identities have interesting applications. Since $Z_n^\lambda(\langle x, y \rangle)$, when $\lambda = \frac{d-2}{2}$, is the reproducing kernel of spherical harmonics of degree n on the unit sphere \mathbb{S}^{d-1} , the identity (1.4) can be used to derive a closed formula for the reproducing kernels of orthogonal polynomials with respect to $W_{\lambda,\mu}(x) = \|x\|^{2\lambda}(1-\|x\|^2)^{\mu-\frac{1}{2}}$ on the unit ball, which is known only in the case $\lambda = 0$ previously. Such a formula plays an essential role for studying Fourier orthogonal expansions. We shall use it, as an application, to determine the critical index for Cesàro means of orthogonal expansion with respect to $W_{\lambda,\mu}$ in Subsection 3.3.

The paper is organized as follows. We prove the main theorems in the following section and discuss applications of the main results in orthogonal polynomials of several variables in Section 3.

2. PROOF OF THEOREMS 1.1 AND 1.2

The proof of Theorem 1 uses multinomial theorem.

Proof of Theorem 1.1. Observe that

$$|r\langle x, u \rangle| \leq r \sum_{i=1}^d |x_i| u_i < \sum_{i=1}^d u_i = 1,$$

so that one can use the multinomial theorem

$$(1 - s_1 - \cdots - s_d)^\nu = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(-\nu)_{|\mathbf{n}|}}{n_1! \cdots n_d!} s_1^{n_1} \cdots s_d^{n_d}$$

(see, e.g., [7, Eq. (220) on p. 329]). We then have

$$\begin{aligned} \int_{\mathcal{T}^d} \frac{1}{(1 - r\langle x, u \rangle)^{|\lambda|}} \prod_{i=1}^d u_i^{\lambda_i-1} du \\ = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(|\lambda|)_{|\mathbf{n}|}}{n_1! \cdots n_d!} r^{|\mathbf{n}|} x_1^{n_1} \cdots x_d^{n_d} \int_{\mathcal{T}^d} \prod_{i=1}^d u_i^{\lambda_i+n_i-1} du. \end{aligned}$$

The integral is now a multivariate beta integral

$$\begin{aligned} \int_{\mathcal{T}^d} \prod_{i=1}^d u_i^{\lambda_i+n_i-1} du &= \int_{T^{d-1}} u_1^{\lambda_1+n_1-1} \cdots u_{d-1}^{\lambda_{d-1}+n_{d-1}-1} (1 - |u|)^{\lambda_d+n_d-1} du_1 \cdots du_{d-1} \\ &= \frac{\Gamma(\lambda_1 + n_1) \cdots \Gamma(\lambda_d + n_d)}{\Gamma(|\lambda| + |\mathbf{n}|)}, \end{aligned}$$

so that

$$\begin{aligned} \int_{\mathcal{T}^d} \frac{1}{(1 - r\langle x, u \rangle)^{|\lambda|}} \prod_{i=1}^d u_i^{\lambda_i-1} du \\ = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(|\lambda|)_{|\mathbf{n}|}}{\Gamma(|\lambda| + |\mathbf{n}|)} \frac{\Gamma(\lambda_1 + n_1) \cdots \Gamma(\lambda_d + n_d)}{n_1! \cdots n_d!} r^{|\mathbf{n}|} x_1^{n_1} \cdots x_d^{n_d}. \end{aligned}$$

Now $(|\lambda|)_{|\mathbf{n}|} = \Gamma(|\lambda| + |\mathbf{n}|)/\Gamma(|\lambda|)$, hence the multiple sum factors into d single sums, and by the binomial theorem

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (rx)^n = \frac{1}{(1 - rx)^\lambda}, \quad |rx| < 1,$$

the result in (1.1) follows. \square

The above proof is communicated to us by Walter Van Assche. Our original proof uses the generalized Gegenbauer polynomials that are orthogonal with respect to the weight function

$$w_{\lambda,\mu}(x) := |x|^{2\mu} (1 - x^2)^{\lambda-\frac{1}{2}}, \quad -1 \leq x \leq 1, \quad \lambda, \mu > -1/2.$$

Let $D_n^{(\lambda,\mu)}$ denote the orthonormal polynomial of degree n with respect to $w_{\lambda,\mu}$,

$$(2.1) \quad c_{\lambda,\mu} \int_{-1}^1 D_n^{(\lambda,\mu)}(x) D_m^{(\lambda,\mu)}(x) w_{(\lambda,\mu)}(x) dx = \delta_{n,m},$$

where $c_{\lambda,\mu}$ is the constant defined by $c_{\lambda,\mu} \int_{-1}^1 w_{\lambda,\mu}(x)dx = 1$. The following identity is used in the original proof and it will also be needed in Section 3.3.

Proposition 2.1. *For $\lambda > 0$ and $\mu > 0$,*

$$(2.2) \quad C_n^{\lambda+\mu}(\cos \theta \cos \phi t + \sin \theta \sin \phi s) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k+j=n-2m} b_{k,j,n}^{\lambda,\mu} (\cos \theta \cos \phi)^k \\ \times (\sin \theta \sin \phi)^j D_{n-k-j}^{(\lambda+j,\mu+k)}(\cos \theta) D_{n-k-j}^{(\lambda+j,\mu+k)}(\cos \phi) C_k^{\mu-\frac{1}{2}}(t) C_j^{\lambda-\frac{1}{2}}(s),$$

where

$$b_{k,j,n}^{\lambda,\mu} = \frac{\Gamma(\mu - \frac{1}{2})\Gamma(\lambda - \frac{1}{2})}{\Gamma(\lambda + \mu)} \frac{\Gamma(\lambda + \mu + k + j + 1)}{(n + \lambda + \mu)\Gamma(k + \mu - \frac{1}{2})\Gamma(j + \lambda - \frac{1}{2})}.$$

The identity (2.2) first appeared in [5, p. 242, (4.7)], proved using a group theoretic method, but the constants were not given explicitly there. An analytic proof with explicit constants was given in [10, Theorem 2.3]. Our original proof uses the integral of (2.2) with respect to $w_{\lambda,\mu}(x)dx$ to prove

$$\frac{1}{(1 - 2rs + r^2)^{\lambda+1}(1 - 2rt + r^2)^{\mu+1}} \\ = c_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \int_0^1 \frac{1}{(1 - 2r(yt + (1-y)s) + r^2)^{\lambda+\mu+2}} y^\lambda (1-y)^\mu dy,$$

which is equivalent to the case $d = 2$ of (1.1) and the case $d > 2$ follows from induction.

In the case of $\lambda_i = 1$ for all $1 \leq i \leq d$, another elementary proof of (1.1) can be deduced from the Hermite-Genocchi formula

$$(2.3) \quad [x_1, \dots, x_d]f = \int_{\mathcal{T}^d} f^{(d-1)}(x_1 t_1 + \dots + x_d t_d) dt,$$

where $[x_1, \dots, x_d]f$ denotes the divided difference of f .

Proof of Theorem 1.2. From (1.1) with $d = 2$, it follows that

$$\frac{1}{(1 - 2rx + r^2)^\lambda (1 - 2ry + r^2)^\mu} = \sigma_{\lambda,\mu} \int_0^1 \frac{s^{\lambda-1} (1-s)^{\mu-1} ds}{(1 - 2r(sx + (1-s)y) + r^2)^{\lambda+\mu}}$$

for $0 \leq r < 1$. Integrating with respect to $(1-y^2)^{\mu-1/2}$, we obtain, by the first identity of (1.2) that

$$\frac{1}{(1 - 2rx + r^2)^\lambda} = c_\mu \sigma_{\lambda,\mu} \int_{-1}^1 \int_0^1 \frac{s^{\lambda-1} (1-s)^{\mu-1} (1-y^2)^{\mu-\frac{1}{2}} ds dy}{(1 - 2r(sx + (1-s)y) + r^2)^{\lambda+\mu}}.$$

Expanding both sides as power series of r , by the first identity of (1.2), the identity (1.3) follows from comparing the coefficients of r^n .

Now, replacing λ by $\lambda + 1$ in the last identity and multiplying by $1 - r^2$, we obtain

$$\frac{1 - r^2}{(1 - 2rx + r^2)^{\lambda+1}} = c_\mu \sigma_{\lambda+1,\mu} \int_{-1}^1 \int_0^1 \frac{(1 - r^2) s^\lambda (1-s)^{\mu-1} (1-y^2)^{\mu-\frac{1}{2}} ds dy}{(1 - 2r(sx + (1-s)y) + r^2)^{\lambda+\mu+1}}.$$

Expanding both sides as power series of r , by the second identity of (1.2), the identity (1.4) follows from comparing the coefficients of r^n . \square

3. APPLICATION TO ORTHOGONAL POLYNOMIALS OF SEVERAL VARIABLES

In the first subsection we recall basics on orthogonal polynomials of several variables and Fourier expansions in terms of them (cf. [3]). Product Gegenbauer polynomials are discussed in the second subsection and orthogonal polynomials on the unit ball are discussed in the third subsection.

3.1. Orthogonal polynomials of several variables. Let W be a nonnegative weight function on a domain Ω of \mathbb{R}^d , normalized so that $\int_{\Omega} W(x)dx = 1$. Let $\mathcal{V}_n^d(W)$ be the space of orthogonal polynomials of degree n with respect to the inner product

$$\langle f, g \rangle_W := \int_{\Omega} f(x)g(x)W(x)dx.$$

It is known that $r_n^d := \dim \mathcal{V}_n^d = \binom{n+d-1}{n}$. Let $\{P_j^n : 1 \leq j \leq r_n^d\}$ be an orthonormal basis of $\mathcal{V}_n^d(W)$; that is,

$$\int_{\Omega} P_j^n(x)P_k^m(x)W(x)dx = \delta_{j,k}\delta_{n,m}.$$

The Fourier coefficient \hat{f}_j^n of $f \in L^2(W, \Omega)$ is defined by $\hat{f}_j^n := \int_{\Omega} f(x)P_j^n(x)W(x)dx$ and the Fourier orthogonal expansion of $f \in L^2(W, \Omega)$ is defined by

$$f = \sum_{n=0}^{\infty} \text{proj}_n f \quad \text{with} \quad \text{proj}_n f(x) := \sum_{j=0}^n \hat{f}_j^n P_j^n(x).$$

The projection operator $\text{proj}_n : L^2(W, \Omega) \mapsto \mathcal{V}_n^d(W)$ can be written as

$$\text{proj}_n f(x) = \int_{\Omega} f(y)P_n(x, y)W(y)dy \quad \text{with} \quad P_n(x, y) := \sum_{j=1}^{r_n^d} P_j^n(x)P_j^n(y),$$

where $P_n(\cdot, \cdot)$ is the reproducing kernel of $\mathcal{V}_n^d(W)$. For $\delta > 0$, the Cesàro (C, δ) means $S_n^\delta f$ of the Fourier orthogonal expansion is defined by

$$S_n^\delta f := \frac{1}{\binom{n+\delta}{d}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} \text{proj}_k f,$$

which can be written as an integral of f against the Cesàro (C, δ) kernel

$$K_n^\delta(x, y) := \frac{1}{\binom{n+\delta}{d}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} P_k(x, y).$$

To emphasize the dependence on W , we will use notations such as $S_n^\delta(W; f)$ and $K_n^\delta(W; \cdot, \cdot)$ in the rest of this section.

3.2. Product Gegenbauer polynomials on the cube. For $\lambda = (\lambda_1, \dots, \lambda_d)$, $\lambda_i > -\frac{1}{2}$, we consider the product Gegenbauer weight function

$$W_\lambda(x) = W_{\lambda,d}(x) := c_\lambda \prod_{i=1}^d w_{\lambda_i}(x_i), \quad x \in [-1, 1]^d,$$

where $c_\lambda = \prod_{i=1}^d c_{\lambda_i}$. It is easy to see that the product Gegenbauer polynomials are orthogonal polynomials and the reproducing kernel $P(W_\lambda; \cdot, \cdot)$ of $\mathcal{V}_n^d(W_\lambda)$ is given by

$$P(W_\lambda; x, y) = \sum_{|\alpha|=n} \frac{1}{H_n^\lambda} P_\alpha(x) P_\alpha(y) \quad \text{with} \quad P_\alpha(x) := \prod_{i=1}^d C_{\alpha_i}^{\lambda_i}(x_i),$$

where $H_n^\lambda = \prod_{i=1}^d h_{\alpha_i}^{\lambda_i}$. The product formula of the Gegenbauer polynomials states that

$$\frac{C_n^\lambda(x) C_n^\lambda(y)}{C_n^\lambda(1)} = c_{\lambda-\frac{1}{2}} \int_{-1}^1 C_n^\lambda(xy + \sqrt{1-x^2}\sqrt{1-y^2}t)(1-t^2)^{\lambda-1} dt,$$

which implies immediately that

$$K_n^\delta(W_\lambda; x, y) = c_{\lambda-\frac{1}{2}} \int_{[-1,1]^d} K_n^\delta(W_\lambda; z(x, y, t), \mathbf{1}) \prod_{i=1}^d (1-t_i^2)^{\lambda_i-1} dt_i,$$

where $z(x, y, t) := (x_1 y_1 + \sqrt{1-x_1^2}\sqrt{1-y_1^2}t_1, \dots, x_d y_d + \sqrt{1-x_d^2}\sqrt{1-y_d^2}t_d)$. Below we deduce a closed form formula for $P_n(W_\lambda; x, \mathbf{1})$ and the (C, δ) kernel $K_n^\delta(W_\lambda; x, \mathbf{1})$.

Theorem 3.1. *Let $\mathbf{1} = (1, \dots, 1)$. Then,*

$$(3.1) \quad P_n(W_\lambda; x, \mathbf{1}) = \sum_{m=0}^{\min\{\lfloor \frac{n}{2} \rfloor, d-1\}} (-1)^m \binom{d-1}{m} \sigma_\lambda \int_{\mathcal{T}^d} Z_{n-m}^{|\lambda|+d-1}(\langle x, y \rangle) \prod_{i=1}^d y_i^{\lambda_i} dy,$$

where $\sigma_\lambda = \Gamma(|\lambda| + d) / \prod_{i=1}^d \Gamma(\lambda_i + 1)$. Furthermore,

$$K_n^{d-2}(W_\lambda; x, \mathbf{1}) = \frac{1}{\binom{n+d-2}{n}} \sum_{m=0}^{\min\{n, d-1\}} \binom{d-1}{m} \sigma_\lambda \int_{\mathcal{T}^d} Z_{n-m}^{|\lambda|+d-1}(\langle x, y \rangle) \prod_{i=1}^d y_i^{\lambda_i} dy.$$

Proof. The identity (1.1) implies that

$$(3.2) \quad \prod_{i=1}^d \frac{1}{(1-2rx_i+r^2)^{\lambda_i+1}} = \sigma_\lambda \int_{\mathcal{T}^d} \frac{1}{(1-2\langle x, y \rangle+r^2)^{|\lambda|+d}} \prod_{i=1}^d y_i^{\lambda_i} dy.$$

Multiplying by $(1-r^2)^d$ and applying (1.2), we see that the left hand side can be expanded as

$$\prod_{i=1}^d \sum_{n=0}^{\infty} Z_{\alpha_i}^{\lambda_i}(x_i) r^n = \sum_{n=0}^{\infty} P_n(W_\lambda; x, \mathbf{1}) r^n,$$

while the right hand side can be expanded, again by (1.2), as

$$\begin{aligned} (1-r^2)^{d-1} \sum_{n=0}^{\infty} \sigma_{\lambda+1} \int_{\mathcal{T}^d} Z_n^{|\lambda|+d-1}(\langle x, y \rangle) \prod_{i=1}^d y_i^{\lambda_i} dy r^n \\ = \sum_{i=1}^{d-1} (-1)^i \binom{d-1}{i} \sum_{n=0}^{\infty} \sigma_{\lambda+1} \int_{\mathcal{T}^d} Z_n^{|\lambda|+d-1}(\langle x, y \rangle) \prod_{i=1}^d y_i^{\lambda_i} dy r^{n+2i}. \end{aligned}$$

The identity (3.1) follows from comparing the coefficient of r^n .

The second identity follows similarly. We multiply (3.2) by $(1+r)^{d-1}(1-r^2)$ so that the left hand side can be expanded as

$$\frac{1}{(1-r)^{d-1}} \prod_{i=1}^d \sum_{n=0}^{\infty} Z_{\alpha_i}^{\lambda_i}(x_i) r^n = \frac{1}{(1-r)^{d-1}} \sum_{n=0}^{\infty} P_n(W_\lambda; x, \mathbf{1}) r^n = \sum_{n=0}^{\infty} K_n^{d-2}(W_\lambda; x, \mathbf{1}) r^n,$$

while the right hand side can be expanded, again by (1.2), as

$$(1+r)^{d-1} \sum_{n=0}^{\infty} \sigma_{\lambda+1} \int_{\mathcal{T}^d} Z_n^{|\lambda|+d-1}(\langle x, y \rangle) \prod_{i=1}^d y_i^{\lambda_i} dy r^n.$$

Expanding $(1+r)^{d-1}$ in power of r , we can again compare the coefficient of r^n . \square

Observe that the right hand side of (3.1) is a sum of at most d terms. In the case of $\lambda = 0$, a closed formula of $P_n(W_0; x, \mathbf{1})$ was derived in [9] as a divided difference, which can be derived from (3.1) by applying the Hermite-Genocchi formula (2.3).

Let $k_n^\delta(w_\lambda)$ denote the kernel of the (C, δ) means of the Gegenbauer polynomials,

$$k_n^\delta(w_\lambda; x, y) = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} \frac{1}{h_n^\lambda} C_k^\lambda(x) C_k^\lambda(y).$$

The proof of Proposition 3.1 also leads to the following proposition.

Proposition 3.2. *For $\delta > -1$,*

$$\begin{aligned} K_n^{\delta+d-1}(W_\lambda; x, \mathbf{1}) &= \frac{1}{\binom{n+\delta+d-1}{n}} \sum_{m=0}^{\min\{n, d-1\}} \binom{d-1}{m} \binom{n-m+\delta}{n-m} \\ &\quad \times \sigma_{\lambda+1} \int_{\mathcal{T}^d} k_{n-m}^\delta(w_{|\lambda|+d-1}; \langle x, y \rangle, 1) \prod_{i=1}^d y_i^{\lambda_i} dy. \end{aligned}$$

One immediate consequence of the above relation shows that $K_n^\delta(W_\kappa; x, y)$ is non-negative if $\delta \geq 2(|\lambda| + d) - 1$, which was proved in [6] by a different method.

3.3. Orthogonal polynomials on the unit ball. On the ball $\mathbb{B}^d = \{x : \|x\| \leq 1\}$ of \mathbb{R}^d , we consider the weight function

$$W_{\lambda, \mu}(x) := b_{\lambda, \mu} \|x\|^{2\lambda} (1 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \lambda \geq 0, \quad \mu > 0, \quad x \in \mathbb{B}^d,$$

where $b_{\lambda, \mu}$ is a constant so that $b_{\lambda, \mu} \int_{\mathbb{B}^d} W_{\lambda, \mu}(x) dx = 1$. Let \mathcal{H}_m^d be the space of spherical harmonics of degree m in d variables. Let $\sigma_m^d := \dim \mathcal{H}_m^d$ and let $\{Y_\nu^m : 1 \leq \nu \leq \sigma_m^d\}$ be an orthonormal basis of \mathcal{H}_m^d . Define

$$P_{j, \nu}^n(x) := P_n^{\left(\mu - \frac{1}{2}, n - 2j + \lambda + \frac{d-2}{2}\right)}(2\|x\|^2 - 1) Y_\nu^{n-2j}(x),$$

where $P_n^{(\alpha, \beta)}(t)$ denotes the usual Jacobi polynomial of degree n .

Proposition 3.3. *The set $\{P_{j, \nu}^n : 1 \leq \nu \leq \sigma_{n-2j}^d, 0 \leq j \leq n/2\}$ is a mutually orthogonal basis of $\mathcal{V}_n^d(W_{\lambda, \mu})$ and the norm of $P_{j, \nu}^n$ in $L^2(W_{\lambda, \mu}, \mathbb{B}^d)$ is given by*

$$H_j^n := \frac{(\lambda + \frac{d}{2})_{n-j} (\mu + \frac{1}{2})_j (n-j+\lambda+\mu+\frac{d-1}{2})}{j! (\lambda + \mu + \frac{d+1}{2})_{n-j} (n+\lambda+\mu+\frac{d-1}{2})},$$

where $(a)_n$ denotes the Pochhammer symbol, $(a)_n := a(a+1) \cdots (a+n-1)$.

Using the spherical polar coordinates $x = rx'$, $0 \leq r \leq 1$ and $x' \in \mathbb{S}^{d-1}$, the case $\lambda = 0$ was worked out explicitly in [3] and the general case follows similarly.

In terms of this basis, the reproducing kernel $P_n(W_{\lambda, \mu}; \cdot, \cdot)$ can be written as

$$(3.3) \quad P_n(W_{\lambda, \mu}; x, y) = \sum_{0 \leq j \leq n/2} \sum_{\nu=1}^{\sigma_{n-2j}^d} \frac{1}{H_{j, n}} P_{j, \nu}^n(x) P_{j, \nu}^n(y).$$

Our main result in this section is the following closed form of this kernel.

Theorem 3.4. *For $\lambda > 0$ and $\mu > 0$,*

$$(3.4) \quad P_n(W_{\lambda,\mu}; x, y) = a_{\lambda,\mu} \int_{-1}^1 \int_0^1 \int_{-1}^1 Z_n^{\lambda+\mu+\frac{d-1}{2}}(\zeta(x, y, u, v, t))(1-t^2)^{\mu-1} dt \\ \times u^{\lambda-1}(1-u)^{\frac{d-2}{2}} du (1-v^2)^{\lambda-\frac{1}{2}} dv,$$

where $a_{\lambda,\mu}$ is a constant such that the integral is 1 if $n = 0$ and

$$\zeta(x, y, u, v, t) := \|x\| \|y\| uv + \langle x, y \rangle (1-u) + \sqrt{1-\|x\|^2} \sqrt{1-\|y\|^2} t;$$

furthermore, if $\lambda > 0$ and $\mu = 0$, then

$$(3.5) \quad P_n(W_{\lambda,0}; x, y) = a_{\lambda,0} \int_{-1}^1 \int_0^1 \frac{1}{2} \left[Z_n^{\lambda+\frac{d-1}{2}}(z(x, y, u, v, 1)) + Z_n^{\lambda+\frac{d-1}{2}}(z(x, y, u, v, -1)) \right] \\ \times u^{\lambda-1}(1-u)^{\frac{d-2}{2}} du (1-v^2)^{\lambda-\frac{1}{2}} dv.$$

Proof. Integrating (2.2) with respect to $w_{\mu-1}(t)dt$, then setting $\cos \theta = \sqrt{1-\|x\|^2}$ and $\cos \phi = \sqrt{1-\|y\|^2}$, and replacing λ by $\lambda + \frac{d-1}{2}$, we deduce that

$$(3.6) \quad c_{\mu-\frac{1}{2}} \int_{-1}^1 C_n^{\lambda+\mu+\frac{d-1}{2}}(\|x\| \|y\| s + \sqrt{1-\|x\|^2} \sqrt{1-\|y\|^2} t) (1-t^2)^{\mu-1} dt \\ = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{0,n-2j,n}^{\lambda+\frac{d-1}{2},\mu} \|x\|^{n-2j} \|y\|^{n-2j} D_{2j}^{(n-2j+\lambda+\frac{d-1}{2},\mu)}(\sqrt{1-\|x\|^2}) \\ \times D_{2j}^{(n-2j+\lambda+\frac{d-1}{2},\mu)}(\sqrt{1-\|y\|^2}) C_{n-2j}^{\lambda+\frac{d-2}{2}}(s).$$

Let $x' = x/\|x\|$ and $y' = y/\|y\|$. By the addition formula of spherical harmonics,

$$\sum_{\nu=1}^{\sigma_m^d} Y_\nu(x) Y_\nu(y) = \|x\|^m \|y\|^m Z_m^{\frac{d-1}{2}}(\langle x', y' \rangle).$$

Furthermore, by [10, (2.1a)], we can deduce that

$$P_j^{(\mu-\frac{1}{2}, n-2j+\lambda+\frac{d-2}{2})}(2\|x\|^2 - 1) = (-1)^j \sqrt{B_{j,n}} D_{2j}^{(n-2j+\lambda+\frac{d-1}{2},\mu)}(\sqrt{1-\|x\|^2}),$$

where

$$B_{j,n} = \frac{\Gamma(n-2j+\lambda+\mu+\frac{d+1}{2})\Gamma(j+\mu+\frac{1}{2})\Gamma(n-j+\lambda+\frac{d}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(n-2j+\lambda+\frac{d}{2})\Gamma(n-j+\lambda+\mu+\frac{d-1}{2})j!(n+\lambda+\mu+\frac{d-1}{2})}.$$

Substituting these two identities into the expression (3.3), we obtain

$$P_n(W_{\lambda,\mu}; x, y) = \sum_{0 \leq j \leq n/2} \frac{B_{j,n}}{H_j^n} D_{2j}^{(n-2j+\lambda+\frac{d-1}{2},\mu)}(\sqrt{1-\|x\|^2}) \\ \times D_{2j}^{(n-2j+\lambda+\frac{d-1}{2},\mu)}(\sqrt{1-\|y\|^2}) \|x\|^{n-2j} \|y\|^{n-2j} Z_{n-2j}^{\frac{d-2}{2}}(\langle x', y' \rangle),$$

in which the last term can be replaced, according to (1.4), by

$$Z_{n-2j}^{\frac{d-2}{2}}(\langle x', y' \rangle) = c \int_{-1}^1 \int_0^1 Z_{n-2j}^{\lambda+\frac{d-2}{2}}(uv + (1-u)\langle x', y' \rangle) u^{\lambda-1}(1-u)^{\frac{d-2}{2}} du w_\lambda(v) dv.$$

Now, a tedious verification shows that the constant

$$\frac{B_{j,n}}{H_j^n} \frac{n-2j+\lambda+\frac{d-2}{2}}{\lambda+\frac{d-2}{2}} = \frac{n+\lambda+\mu+\frac{d-1}{2}}{\lambda+\mu+\frac{d-1}{2}} b_{0,n-2j,n}^{\lambda+\frac{d-1}{2},\mu}.$$

Consequently, (3.4) follows from (3.6). Finally, (3.5) follows from (3.4) by taking the limit $\mu \rightarrow 0$. \square

Corollary 3.5. *For $\lambda > 0$ and $\mu > 0$,*

$$(3.7) \quad P_n(W_{\lambda,\mu}; x, 0) = D_n^{(\lambda+\frac{d-1}{2}, \mu)}(1) D_n^{(\lambda+\frac{d-1}{2}, \mu)}(\sqrt{1-\|x\|^2}),$$

where $D_n^{(\lambda,\mu)}$ is the generalized Gegenbauer polynomial defined in (2.1).

Proof. Setting $y = 0$ in (3.4) shows that

$$P_n(W_{\lambda,\mu}; x, 0) = c_{\mu-\frac{1}{2}} \int_{-1}^1 Z_n^{\lambda+\mu+\frac{d-1}{2}}(\sqrt{1-\|x\|^2}t) (1-t^2)^{\mu-1} dt,$$

from which the stated result follows from [10, (2.11)]. \square

Taking the limit $\lambda = 0$, the triple integrals of (3.4) becomes one layer, the resulted identity was first proved in [11], which plays an essential role in the study of Fourier orthogonal expansions with respect to the classical weight function $W_{0,\mu}$ on \mathbb{B}^d . The closed form formula in Theorem 3.4 should play a similar role. We give one application.

For $f \in L^1(w_{\lambda+\mu+\frac{d-1}{2}}; [-1, 1])$ and $x, y \in \mathbb{B}^d$, define

$$G_x f(y) := a_{\lambda,\mu} \int_{-1}^1 \int_0^1 \int_{-1}^1 f(\zeta(x, y, u, v, t)) (1-t^2)^{\mu-1} dt u^{\lambda-1} (1-u)^{\frac{d-2}{2}} du w_\lambda(v) dv.$$

As a consequence of Theorem 3.4, we can write

$$(3.8) \quad K_n^\delta(W_{\lambda,\mu}; x, y) = G_x \left[k_n^\delta(w_{\lambda+\mu+\frac{d-1}{2}}; \cdot, 1) \right] (y),$$

where $k_n^\delta(w_\lambda; s, t)$ denotes the Cesàro (C, δ) means of the Gegenbauer series.

Theorem 3.6. *For $\lambda \geq 0$ and $\mu \geq 0$, the Cesàro (C, δ) means for $W_{\lambda,\mu}$ satisfy*

1. *if $\delta \geq 2\lambda + 2\mu + d$, then $S_n^\delta(W_{\lambda,\mu}; f) \geq 0$ if $f(x) \geq 0$;*
2. *$S_n^\delta(W_{\lambda,\mu}; f)$ converge to f in $L^1(W_{\lambda,\mu}; \mathbb{B}^d)$ norm or $C(\mathbb{B}^d)$ norm if and only if $\delta > \lambda + \mu + \frac{d-1}{2}$.*

Proof. The first assertion follows immediately from the non-negativity of the Gegenbauer series [4]. To prove the second assertion, we first show that

$$(3.9) \quad b_{\lambda,\mu} \int_{\mathbb{B}^d} G_x g(y) W_{\lambda,\mu}(y) dy = c_{\lambda+\mu+\frac{d-1}{2}} \int_{-1}^1 g(t) w_{\lambda+\mu+\frac{d-1}{2}}(t) dt$$

for $g \in L^1(w_{\lambda+\mu+\frac{d-1}{2}}; [-1, 1])$. It suffices to prove it for g being a polynomial, which

we can write as $g(t) = \sum_{k=0}^N \hat{g}_k Z_k^{\lambda+\mu+\frac{d-1}{2}}(t)$, where \hat{g}_0 is exactly the right hand side of (3.9). By Theorem 3.4, $G_x g(y) = \sum_{k=0}^N \hat{g}_k P_k(W_{\lambda,\mu}; x, y)$, so that (3.9) follows from the orthogonality of $P_k(W_{\lambda,\mu}; x, \cdot)$. A standard argument shows that $S_n^\delta(W_{\lambda,\mu}; f)$ converges to f in either $L^1(W_{\lambda,\mu}; \mathbb{B}^d)$ norm or $C(\mathbb{B}^d)$ norm if, and only if,

$$\Lambda_n(x) := b_{\lambda,\mu} \int_{\mathbb{B}^d} |K_n^\delta(W_{\lambda,\mu}; x, y)| W_{\lambda,\mu}(y) dy$$

is bounded, independent of n , for all $x \in \mathbb{B}^d$. From (3.8) and (3.9), we conclude that

$$\Lambda_n(x) \leq c_{\lambda+\mu+\frac{d-1}{2}} \int_{-1}^1 \left| k_n^\delta(w_{\lambda+\mu+\frac{d-1}{2}}; t, 1) \right| w_{\lambda+\mu+\frac{d-1}{2}}(t) dt,$$

which is finite if $\delta > \lambda + \mu + \frac{d-1}{2}$ (Theorem 9.1.32 [8, p. 246]). Furthermore, by (3.7), we obtain

$$K_n^\delta(W_{\lambda,\mu}; 0, y) = k_n^\delta\left(w_{\lambda+\frac{d-1}{2},\mu}; \sqrt{1-\|x\|^2}, 1\right),$$

where $w_{\lambda+\frac{d-1}{2},\mu}$ is the generalized Gegenbauer weight. Using spherical-polar coordinate and making a change of variable, it is easy to see that

$$b_{\lambda,\mu} \int_{\mathbb{B}^d} |K_n^\delta(W_{\lambda,\mu}; 0, y)| W_{\lambda,\mu}(y) dy = c_{\lambda+\frac{d-1}{2},\mu} \int_{-1}^1 \left| k_n^\delta(w_{\lambda+\frac{d-1}{2},\mu}; 1, t) \right| w_{\lambda+\frac{d-1}{2},\mu}(t) dt,$$

which is bounded if, and only if, $\delta > \lambda + \mu + \frac{d-1}{2}$ by [2, Theorem 2.4]. \square

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APPENDIX. ALTERNATIVE PROOF OF THEOREM 1.1

This appendix contains the original proof of Theorem 1.1, which shows how the identity was discovered. The proof is added here in respond to a request by a reader and it will appear only in the ArXiv version of this paper.

Proof of Theorem 1.1. Setting $x = \cos \theta = \cos \phi$ in (2.2) and integrating with respect to $w_{\lambda,\mu}$, we obtain by the orthonormality of $D_n^{(\lambda+j,\mu+k)}$ that

$$\begin{aligned} c_{\lambda,\mu} \int_{-1}^1 C_n^{\lambda+\mu}(x^2 t + (1-x^2)s) w_{\lambda,\mu}(x) dx \\ = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k+j=n-2m} b_{k,j}^n \frac{c_{\lambda,\mu}}{c_{\lambda+j,\mu+k}} C_k^{\mu-\frac{1}{2}}(t) C_j^{\lambda-\frac{1}{2}}(s). \end{aligned}$$

Changing variable $y = x^2$ in the integral and simplifying the constants in the right hand side, the above identity can be written as

$$c_{\lambda,\mu} \int_0^1 Z_n^{\lambda+\mu}(yt + (1-y)s) y^{\lambda-\frac{1}{2}}(1-y)^{\mu-\frac{1}{2}} dy = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k+j=n-2m} Z_k^{\mu-\frac{1}{2}}(t) Z_j^{\lambda-\frac{1}{2}}(s).$$

In particular, using the identity

$$\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k+j=n-2m} a_{k,j} + \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k+j=n-1-2m} a_{k,j} = \sum_{m=0}^n \sum_{k+j=m} a_{k,j},$$

and replacing λ by $\lambda + 1/2$ and μ by $\mu + 1/2$, we deduce that

$$\begin{aligned} \sum_{m=0}^n \sum_{k+j=m} Z_k^{\mu}(t) Z_j^{\lambda}(s) \\ = c_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \int_0^1 \left[Z_n^{\lambda+\mu+1}(yt + (1-y)s) + Z_{n-1}^{\lambda+\mu+1}(yt + (1-y)s) \right] y^{\lambda}(1-y)^{\mu} dy. \end{aligned}$$

Next we multiply the above identity by r^n , $0 \leq r < 1$, and summing up over n . In the left hand side, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k+j=m} Z_k^{\mu}(t) Z_j^{\lambda}(s) r^n &= \frac{1}{1-r} \sum_{n=0}^{\infty} \sum_{k+j=n} Z_k^{\mu}(t) Z_j^{\lambda}(s) r^n \\ &= \frac{1}{1-r} \sum_{k=0}^{\infty} Z_k^{\mu}(t) r^k \sum_{j=0}^{\infty} Z_j^{\lambda}(s) r^j = \frac{(1+r)(1-r^2)}{(1-2rs+r^2)^{\lambda+1}(1-2rt+r^2)^{\mu+1}} \end{aligned}$$

by (1.2). The right hand side can be summed up by using

$$\sum_{n=0}^{\infty} \left[Z_n^{\lambda+\mu+1}(u) + Z_{n-1}^{\lambda+\mu+1}(u) \right] r^n = (1+r) \sum_{n=0}^{\infty} Z_n^{\lambda+\mu+1}(u) r^n = \frac{(1+r)(1-r^2)}{(1-2ru+r^2)^{\lambda+\mu+2}}.$$

Putting these together, we have proved that

$$\begin{aligned} \frac{1}{(1-2rs+r^2)^{\lambda+1}(1-2rt+r^2)^{\mu+1}} \\ = c_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \int_0^1 \frac{1}{(1-2r(yt + (1-y)s) + r^2)^{\lambda+\mu+2}} y^{\lambda}(1-y)^{\mu} dy. \end{aligned}$$

Rescaling and replacing $2r/(1+r^2)$ by r , it follows that

$$\frac{1}{(1-rs)^{\lambda+1}(1-rt)^{\mu+1}} = c_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \int_0^1 \frac{1}{(1-r(yt + (1-y)s))^{\lambda+\mu+2}} y^{\lambda}(1-y)^{\mu} dy,$$

which proves (1.1) for $d = 2$ and $\lambda, \mu > 1/2$ when we replace λ by $\lambda - 1$ and μ by $\mu - 1$. Analytic continuation shows that the identity holds for $\lambda, \mu > 0$.

The general case of (1.1) follows from induction. Assume that (1.1) has been established for d variables. Let the constant in front of the integral in (1.1) be denoted by σ_{λ} , its value can be determined by setting $r = 0$ and is not important for the induction. For $x \in \mathbb{R}^{d+1}$, write $x = (x_1, x')$ and $\lambda = (\lambda_1, \lambda')$. Then, using (1.1) for $d = 2$,

$$\begin{aligned} \prod_{i=1}^{d+1} \frac{1}{(1 - rx_i)^{\lambda_i}} &= \sigma_{\lambda'} \frac{1}{(1 - rx_1)^{\lambda_1}} \int_{\mathcal{T}^d} \frac{\prod_{i=2}^{d+1} u_i^{\lambda_i-1}}{(1 - r\langle x', u \rangle)^{|\lambda'|}} du \\ &= c \int_{\mathcal{T}^d} \int_0^1 \frac{\prod_{i=2}^{d+1} u_i^{\lambda_i-1} (1 - t_1)^{|\lambda'|} t_1^{\lambda_1}}{(1 - ((1 - t_1)\langle x', u \rangle + t_1 u_1)r)^{|\lambda|}} dt_1 du, \end{aligned}$$

where we have written $u = (u_2, \dots, u_{d+1}) \in \mathcal{T}^d$. Since $|u| = 1$, we write $u_{d+1} = 1 - u_2 - \dots - u_d$ and make a change of variables $t_i = (1 - t_1)u_i$ for $i = 2, \dots, d$, it follows that

$$(1 - t_1)^{|\lambda'|} t_1^{\lambda_1} \prod_{i=2}^{d+1} u_i^{\lambda_i} = \prod_{i=1}^d t_i^{\lambda_i} (1 - t_1 - \dots - t_{d+1})^{\lambda_{d+1}} = \prod_{i=2}^{d+1} t_i^{\lambda_i},$$

where $t_{d+1} = 1 - t_1 - \dots - t_d$ and, moreover, $(1 - t_1)\langle x', u \rangle + t_1 u_1 = \langle x, t \rangle$ with $x = (x_1, x')$, $t = (t_1, \dots, t_{d+1})$ and $t_{d+1} = 1 - t_1 - \dots - t_d$. Consequently, since $(1 - t_1)^{d-1} du = dt_2 \dots dt_d$, we conclude that

$$\prod_{i=1}^{d+1} \frac{1}{(1 - rx_i)^{\lambda_i}} = c \int_0^1 \int_{\mathcal{T}_{1-t_1}^d} \frac{\prod_{i=1}^{d+1} t_i^{\lambda_i-1}}{(1 - \langle x, t \rangle r)^{|\lambda|}} dt_1 \dots dt_{d+1} = c \int_{\mathcal{T}^{d+1}} \frac{\prod_{i=1}^{d+1} t_i^{\lambda_i-1}}{(1 - \langle x, t \rangle r)^{|\lambda|}} dt,$$

where c is a constant and its value can be determined by setting $r = 0$. This completes the proof of (1.1). \square

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